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Infinite Goldie Dimensions

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0. INTRODUCTION

Let R be a ring with identity and M a unital right R-module. By the finite Goldie dimension of M is meant the largest integer m (provided it exists) such that M contains the direct sum of m nonzero submodules. If no such m exists, then M is said to have infinite Goldie dimension.

Until recently, not much attention has been given to the case of infinite Goldie dimensions, though the definition is immediate. The *Goldie dimension* of M—denoted Gd M—is the supremum λ of all cardinals κ such that M contains the direct sum of κ nonzero submodules. We believe that one of the main reasons why infinite Goldie dimensions have been ignored is that suprema are difficult to handle, and there is no guarantee that a module M of Goldie dimension λ contains the direct sum of exactly λ nonzero submodules.

Given a cardinal number κ, we say κ is *attained in M* if M contains a direct sum of κ nonzero submodules. If κ is not a limit cardinal, i.e., if it is of the form $\kappa = \aleph_{\alpha+1}$ for some ordinal α, then $\kappa \leqq \text{Gd } M$ is attained in M.

Recall that an infinite cardinal κ is called *regular* if $\kappa_i < \kappa$ for $i \in I$ with $|I| < \kappa$ implies $\sum \kappa_i < \kappa$. Otherwise it is called *singular*. An uncountable, regular, limit cardinal is said to be *inaccessible* ([HJ, p. 163] or [L, p. 137]). The reader is reminded that the existence of inaccessible cardinals cannot be proved in ZFC (Zermelo–Fraenkel set theory with Axiom of Choice added), and that in the constructible universe, there are no such cardinals. All proofs here are within the framework of ZFC.

Our purpose here is to show that if Gd M is not an inaccessible cardinal, then Gd M is attained in M. Our result is best possible; for inaccessible cardinals see the remark at the end.

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The paper [D] explores some ring and module theoretic significance and applications of infinite Goldie dimensions.

1. Preliminaries

It will be assumed that the ring R considered has an identity and that all the modules are right unital over R. The symbols $\leqq$ and $<$ stand for submodule and proper submodule, respectively. The injective hull of the module M is denoted by $E(M) = EM$. The symbol $|X|$ means the cardinality of X, and cof κ denotes the cofinality of the cardinal κ, i.e., the least cardinal (or ordinal) β such that κ can be represented in the form $\kappa = \sum \{\kappa_i \mid i \in I\}$ with all $\kappa_i < \kappa$ and $|I| = \beta$.

The Goldie dimension Gd M of M was defined in the introduction. It is evident that Gd $0 = 0$, and if N is an essential extension of M, then Gd $N =$ Gd M; in particular, Gd $EM =$ Gd M. Thus it suffices to consider only injective modules in studying the question when the Goldie dimension is attained.

Our proof relies heavily on the exchange property of injectives. For the next lemma, see [W, p. 265].

LEMMA 1. *Let B_j ($j \in J$) be injective submodules of the injective R-module M such that $M = E(\oplus\{B_j \mid j \in J\})$. Given a summand N of M, there exist submodules $C_j \leqq B_j$ ($j \in J$) such that*

(i) $M = N \oplus E(\oplus\{C_j \mid j \in J\})$;

(ii) $B_j = C_j \oplus D_j$ *for suitable* $D_j \leqq B_j$ ($j \in J$);

(iii) $N \cong E(\oplus\{D_j \mid j \in J\})$. ∎

The next lemma is crucial in the proof.

LEMMA 2. *Assume that M is an essential extension of a direct sum $\oplus\{A_i \mid i \in I\}$ of submodules, and let κ be an infinite cardinal with* cof $\kappa > |I|$. *If κ is attained in M, then there exists an $n \in I$ such that κ is attained in A_n (which, in particular, entails that* Gd $A_n \geqq \kappa$).

Proof. It may be assumed that $M = E(\oplus\{A_i \mid i \in I\})$ and that all the A_i are injective. Suppose that κ is attained in M; i.e., M is of the form

$$M = E\left(\bigoplus_{j \in J} B_j\right),$$

where $|J| = \kappa$, and the B_j are nonzero injective modules.

First, suppose $I = \{1, ..., m\}$ is a finite index set. From Lemma 1 it follows via a straightforward induction that each $A_i \cong E(\oplus\{C_{ij} \mid j \in J\})$ for suitable submodules C_{ij} satisfying $B_j = C_{1j} \oplus \cdots \oplus C_{mj}$. For every $j \in J$, one of $C_{1j}, ..., C_{mj}$ is nontrivial, so there is an $n \leqq m$ such that the index set $\{j \in J \mid C_{nj} \neq 0\}$ has the same cardinality as J.

Next assume I is infinite. Then the set $\mathscr{I}$ of all finite subsets of I has the same cardinality as I. For $F \in \mathscr{I}$, define

$$A_F = \oplus\{A_i \mid i \in F\} \qquad \text{and} \qquad J_F = \{j \in J \mid A_F \cap B_j \neq 0\}.$$

Then $\bigcup \{J_F \mid F \in \mathscr{I}\} = J$ implies, in view of the hypothesis on cof $|J|$, that there exists an $F \in \mathscr{I}$, say $F = \{1, ..., m\}$, such that $|J_F| = |J|$. Apply Lemma 1 to $N = A_F$ to conclude the existence of submodules $D_j \leqq B_j$ $(j \in J)$ satisfying

$$A_F \cong E(\oplus\{D_j \mid j \in J\}).$$

Manifestly, $j \in J_F$ implies $D_j \neq 0$. Thus we are now back in the case of a finite index set I which has been settled in the preceding paragraph. ▌

Our next purpose is to verify the additivity of the Goldie dimension. The following theorem holds even if $\mathrm{Gd}(\oplus_{i \in I} A_i)$ is an inaccessible cardinal.

THEOREM 3. $\mathrm{Gd}(\oplus_{i \in I} A_i) = \sum_{i \in I} \mathrm{Gd}\, A_i$.

Proof. If not only $|I|$ is finite, but all the $\mathrm{Gd}\, A_i$ $(i \in I)$ are finite as well, then the above theorem is known. We may thus assume that one of the above listed cardinals is infinite and of course all $\mathrm{Gd}\, A_i > 0$. Hence

$$\sum_{i \in I} \mathrm{Gd}\, A_i = \max\left(|I|, \sup_{i \in I} \mathrm{Gd}\, A_i\right)$$

is an infinite cardinal $\aleph_\alpha$ ([HJ, p. 156, 2.3 and 2.4] or [L, p. 103, 4.4]).

Set $M = \oplus\{A_i \mid i \in I\}$. Since $|I| < \mathrm{Gd}\, M$ and $\mathrm{Gd}\, A_i < \mathrm{Gd}\, M$ for all i, $\mathrm{Gd}\, M \geqq \aleph_\alpha$ is clear. For the reverse inequality, suppose that $\aleph_{\alpha+1} \leqq \mathrm{Gd}\, M$. Since $\aleph_{\alpha+1}$ is then attained in M, and since $|I| \leqq \aleph_\alpha < \aleph_{\alpha+1} = \mathrm{cof}(\aleph_{\alpha+1})$, we infer from Lemma 2 that $\aleph_{\alpha+1}$ is attained in A_n for some $n \in I$. Hence $\mathrm{Gd}\, A_n \geqq \aleph_{\alpha+1}$, a contradiction. ▌

The next lemma gives a sufficient condition for the attainment of the Goldie dimension.

LEMMA 4. *If each nonzero submodule U of a module M contains, in turn, a nonzero submodule V with* $\mathrm{Gd}\, V < \mathrm{Gd}\, M$, *then* $\mathrm{Gd}\, M$ *is attained in M.*

Proof. Clearly, we may assume that $\kappa = \mathrm{Gd}\, M$ is an infinite limit cardinal. Write $\kappa = \sup\{\lambda(\alpha) \mid \alpha < \beta\}$, where $\beta = \mathrm{cof}\, \kappa$ and the $\lambda(\alpha)$ are nonlimit cardinals strictly below κ.

Let $\{A_i \mid i \in I\}$ be a maximal independent family of nonzero submodules of M with the property that Gd $A_i < \kappa$ for all i. By hypothesis, $\oplus\{A_i \mid i \in I\}$ is essential in M, and we infer that

$$\kappa = \mathrm{Gd}\ M = \mathrm{Gd}\left(\bigoplus_{i \in I} A_i\right) = \sum_{i \in I} \mathrm{Gd}\ A_i = \max\left(|I|, \sup_{i \in I} \mathrm{Gd}\ A_i\right).$$

Clearly, $|I| \geqq \beta$. If $|I| = \kappa$, we are done. Otherwise, we have sup Gd $A_i = \kappa$, which allows us to pick a sequence of distinct elements $i(\alpha) \in I$ for each $\alpha < \beta$, such that Gd $A_{i(\alpha)} \geqq \lambda(\alpha)$ for all $\alpha < \beta$. Since each $\lambda(\alpha)$ is attained in $A_{i(\alpha)}$, the cardinal κ is attained in the direct sum of the A_i, and hence in M. ∎

2. Main Result

The first infinite cardinal $\aleph_0$ requires special treatment which, for the sake of completeness, is included here.

LEMMA 5. *The Goldie dimension of a module M with* Gd $M = \aleph_0$ *is attained.*

Proof. Assume M is injective. It is known that a module which has the descending (or ascending) chain condition on direct summands is a direct sum of a finite number of indecomposable modules [AF, p. 128, Proposition 10.14]. This means that there exists a countably infinite properly descending chain of summands $M = M_0 \supset M_1 \supset M_2 \supset \cdots$. Let $M_i = A_i \oplus M_{i+1}$ for some $0 \neq A_i < M$ for all i. Then $\oplus\{A_i \mid i < \omega\} \leqq M$ shows that Gd $M = \aleph_0$ is attained. ∎

Our main result is as follows.

THEOREM 6. *If the Goldie dimension of a module is not an inaccessible cardinal, then it is attained.*

Proof. Suppose that $\kappa =$ Gd M is not attained in M. In view of Lemma 5, cof $\kappa < \kappa$. Lemma 4 guarantees that M contains a nonzero submodule U such that Gd $V = \kappa$ for all nonzero submodules V of U. In particular, Gd $U = \kappa$. Since cof $\kappa <$ Gd U, we may choose an independent family $\{A_i \mid i < \beta\}$ of nonzero submodules where $\beta =$ cof κ. We can now argue, as in the proof of Lemma 4, that κ is attained in the direct sum of the A_i. This contradicts our hypothesis that κ is not attained in M. ∎

From Lemma 4 we derive:

COROLLARY 7. *Given a ring R, there exists a module M whose Goldie dimension $\kappa = \text{Gd } M$ is not attained in M if and only if there exists a right ideal $L < R$ such that $\kappa = \text{Gd } R/L$ is not attained in R/L. If this is the case, then $\kappa \leqslant |R|$.*

It now follows from [AF, p. 315, 28.4(f)] that if R is left perfect, then the Goldie dimension is attained in every right R-module.

3. EXAMPLES

Here Gd R will denote the Goldie dimension of the ring R viewed as a right module over itself. The following two examples show that rings of arbitrary Goldie dimensions κ exist.

EXAMPLE 8. Take any set X of cardinality κ, and any field F with $|F| < \kappa$. Let R be the free F-algebra of all noncommuting polynomials with indeterminates in X. Then Gd $R = |X| = \kappa$ is attained in R. Every principal right ideal of R is R-isomorphic to R. Hence the condition in Lemma 4 is sufficient, but not necessary for the attainment of the Goldie dimension of a module.

EXAMPLE 9. For an infinite initial ordinal β and a ring F with $|F| < \beta$, set $F_\alpha = F$ for all $\alpha < \beta$. Let R be the subring of $\prod\{F_\alpha \mid \alpha < \beta\}$ consisting of eventually constant vectors. Then $|R| = \beta$ and $\oplus\{F_\alpha \mid \alpha < \beta\} \leqq R$ together imply that Gd $R = \beta$.

Remark (added March 5, 1987). We are indebted to S. Shelah for pointing out that for every inaccessible cardinal κ, there is a ring whose Goldie dimension is κ, not attained. In fact, Erdös and Tarski [ET] define, for any inaccessible cardinal κ a Boolean lattice B where the supremum of the cardinalities of sets each consisting of pairwise disjoint elements of B is κ, but B fails to contain κ pairwise disjoint elements. Such a B viewed as a Boolean ring provides an example of a ring whose Goldie dimension is not attained. (Though [ET] solves a problem similar to ours for partially ordered sets, our result is not a consequence of [ET].)

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